S.-T. Yau College Student Mathematics Contests 2025

Geometry and Topology

(6 problems)

Problem 1. Let $n > 1$ be a positive integer.

(i) Is the tangent bundle of $S^{2n} \times S^{2n}$ trivial? Justify your answer.
(ii) Is the tangent bundle of $S^{2n} \times S^{2n-1}$ trivial? Justify your answer.

Problem 2. The Lie group $SU(2)$ is diffeomorphic to the 3-sphere S^3, with Lie algebra spanned by

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

A positive-definite inner product on the Lie algebra determines a left-invariant Riemannian metric on the Lie group. The Berger metrics g_ϵ on S^3 are obtained by making the inner product on the Lie algebra such that $\epsilon^{-1}X_1$, X_2, and X_3 form an orthonormal frame, where $\epsilon > 0$ is a positive number.

(i) Let ∇ be the Levi-Civita connection for g_ϵ. Prove that $\nabla_{X_i} X_i = 0$ for $i = 1, 2, 3$.
(ii) Compute the scalar curvature for g_ϵ.

Problem 3. Let $n \geq 1$ be a positive integer, and let $U(n)$ be the group of unitary matrices in $\mathbb{C}^n$.

(i) Compute $\pi_2(U(n))$ and $\pi_3(U(n))$.
(ii) Prove that the determinant map $\det : U(n) \to S^1$ induces an isomorphism on fundamental groups.

Problem 4. Prove that $\mathbb{CP}^2$ does not admit an immersion into $\mathbb{R}^6$.

Problem 5. Let (M, g) be a closed, oriented, two-dimensional smooth Riemannian manifold with Gaussian curvature K. Suppose u is a smooth function, and define a new metric $\tilde{g} := e^{2u} g$ in the same conformal class as g, with Gaussian curvature $\tilde{K}$.

(i) Prove that u satisfies the equation

$$\Delta u - K + \tilde{K} e^{2u} = 0,$$

where Δ is the Laplace–Beltrami operator defined by the metric g.

(ii) Suppose the Euler characteristic $\chi(M) = 0$. Prove that either $\tilde{K} \equiv 0$ or

$$\int_M \tilde{K} e^{2f} \, d\mathrm{Vol}_g < 0,$$

where f is a solution to the equation $\Delta f = K$.

Problem 6. Let $n \geq 1$ be a positive integer.

(i) Does there exist a self-homeomorphism of $\mathbb{CP}^n$ with no fixed points? Either construct such a homeomorphism or prove that none exists.
(ii) Does there exist a self-homeomorphism of $\mathbb{CP}^2 \times \mathbb{CP}^2$ with no fixed points? Either construct such a homeomorphism or prove that none exists.